# Variational formulation of finite-strain plate model of nematic liquid crystal elastomers 

Kumar Gaurav<br>Roll number: 20105271<br>gmail id: kumgaurav5620@gmail.com<br>IIT Kanpur

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## Introduction

Typical liquids are isotropic materials whose optical, electrical and magnetic properties doesn't depend on the direction. However, liquid crystals are type of fluids which consists of stiff rod-like organic molecules with long range orientational order. These molecules may not exhibit any positional order. There are three types of liquid crystals: nematic, smectic and cholesteric. The average direction along which the longest axis of the rod is aligned is called the director, $\boldsymbol{n}$. In the nematic liquid crystals there is only orientational order and hence director vector is uniform for a long range. Smectic liquid crystals also have additional positional orientation.

Long polymer chains can also contain rigid rod-like structures which can make the molecules anisotropic. These macro molecules are called liquid crystalline polymers (LCP). They have the large elastic deformability of elastomers combined with the properties of liquid crystals. Presence of rigid rods can result in the change of average molecular shape from spherical to spheroidal. In the case of prolate spheroid, longest axis points along the director (see Fig.1). Liquid crystalline elastomers are network of crosslinked polymer chains of liquid crystals. Because of the linking of the polymer chains, their topology is fixed and the melt becomes an elastic rubber like solid [Warner and Terentjev, 2007]. Nematic Liquid Crystal Elastomers (NCLEs) can undergo large reversible deformations and hence has applications in the field of artificial muscles, soft robotics, actuators, reconfigurable structures and shape-shifting structures.

NCLEs shows various interesting phenomena during stretching experiments like stripe domain instability, soft elasticity, director rotation and director jump. To explain these, [Bladon et al., 1993] proposed a neoclassical free energy which was the generalization of classical rubber elasticity. [Anderson et al., 1999] developed a continuum model for NCLEs in which they assumed the strain energy to be a function of deformation gradient, the director and the gradient of director. They suggested a two-constant energy density which combined both neoclassical energy density due to elastic stress and Frank elasticity due to distortion of the director. Recently, [Zhang et al., 2019] also developed a general theory based on continuum mechanics. They obtained the linear momentum balance and the evolution equations of the director based on the dissipation principle, the balance of momentum and. These continuum models can be used to study NLCEs but they involve complex 3D governing equations. However, owing to the fact that the thickness of NLCEs is smaller in comparison to other dimensions, we can treat these structure as an anisotropic and hyperelastic plate. Inspired by this, Foppl-von-Karman-type plate model have been derived. In the latter sections a variational approach is used to derive the 3D equilibrium condition for a NCLE plate. Using a strain energy density function proposed by [Anderson et al., 1999], we solve for the bending of a NCLE-substrate sample.

## The variational formulation

The NCLE plate in its initial (reference) configuration, $\boldsymbol{R}_{0}$, is shown in Fig. 2. A cross section of the plate is represented by $\Omega$. The thickness of the plate, $2 h$, is much smaller than the other two dimensions. This will be required in approximating the 3 D equation to 2 D . The plate occupies the region $\Omega \times[0,2 h]$ in its reference configuration. The current configuration of the plate is denoted by $\boldsymbol{R}_{t}$. The coordinates of a point is denoted by $\boldsymbol{X}(\boldsymbol{r}, Z)$ in $\boldsymbol{R}_{0}$ and $\boldsymbol{x}(\boldsymbol{r}, Z)$ in $\boldsymbol{R}_{t}$, where

$$
\begin{equation*}
\boldsymbol{r}=X_{1} \boldsymbol{E}_{1}+X_{2} \boldsymbol{E}_{2} \tag{1}
\end{equation*}
$$

represents the in-plane coordinates. Using above notations, the deformation gradient tensor becomes,

$$
\begin{equation*}
\boldsymbol{F}=\nabla \boldsymbol{x}(\boldsymbol{r}, Z)=\frac{\partial \boldsymbol{x}(\boldsymbol{r}, Z)}{\partial \boldsymbol{X}}=\frac{\partial \boldsymbol{x}(\boldsymbol{r}, Z)}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{x}(\boldsymbol{r}, Z)}{\partial Z} \otimes \boldsymbol{k}=\bar{\nabla} \boldsymbol{x}(\boldsymbol{r}, Z)+\frac{\partial \boldsymbol{x}(\boldsymbol{r}, Z)}{\partial Z} \otimes \boldsymbol{k} \tag{2}
\end{equation*}
$$

where $\bar{\nabla}$ represents the in-plane gradient. Similarly, the orientation gradient tensor becomes,

$$
\begin{equation*}
\boldsymbol{G}=\nabla \boldsymbol{n}(\boldsymbol{r}, Z)=\frac{\partial \boldsymbol{n}(\boldsymbol{r}, Z)}{\partial \boldsymbol{X}}=\frac{\partial \boldsymbol{n}(\boldsymbol{r}, Z)}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{n}(\boldsymbol{r}, Z)}{\partial Z} \otimes \boldsymbol{k}=\bar{\nabla} \boldsymbol{n}(\boldsymbol{r}, Z)+\frac{\partial \boldsymbol{n}(\boldsymbol{r}, Z)}{\partial Z} \otimes \boldsymbol{k} \tag{3}
\end{equation*}
$$

where $\boldsymbol{n}(\boldsymbol{r}, Z)$ is the director in the current configuration, $\boldsymbol{R}_{t}$. Its counterpart in the reference configuration is denoted by $\boldsymbol{n}_{0}$. The two constraints corresponding to the incompressiblity and the unit length of the director are given as

$$
\begin{align*}
& R_{1}(\boldsymbol{F})=\operatorname{det}(\boldsymbol{F})-1=0  \tag{4}\\
\text { and } & R_{2}(\boldsymbol{n})=\boldsymbol{n} . \boldsymbol{n}-1=0 . \tag{5}
\end{align*}
$$

Taking gradient of (5) gives,

$$
\frac{\partial \boldsymbol{n} . \boldsymbol{n}}{\partial \boldsymbol{X}}=\frac{\partial n_{i} n_{i}}{\partial X_{j}} \boldsymbol{E}_{j}=2 n_{i} \frac{\partial n_{i}}{\partial X_{j}} \boldsymbol{E}_{j}=\boldsymbol{G}^{T} \boldsymbol{n}=0
$$

Note that [Liu et al., 2020] has derived the same expression by differentiating w.r.t $\boldsymbol{n}$ which appears to be wrong or typo in the the text. This can also be equivalently written as,

$$
\begin{equation*}
R_{3}(\boldsymbol{n}, \boldsymbol{G})=\boldsymbol{n} \cdot \boldsymbol{G} \boldsymbol{a}=0 \tag{6}
\end{equation*}
$$

for an arbitrary vector $\boldsymbol{a}$. This is a constraint condition because it involves only independent variables $\boldsymbol{G}$ and $\boldsymbol{n}$ ([Anderson et al., 1999]). We need this additional condition because we will be using $\boldsymbol{G}$ as an independent variable in our formulation. The lateral boundaries of the plate where displacement boundary condition,

$$
\begin{equation*}
\boldsymbol{x}=\boldsymbol{b}(s, Z) \tag{7}
\end{equation*}
$$

where $s$ is the arc length parameter, is specified is denoted by $\boldsymbol{\partial} \boldsymbol{\Omega}_{0}$. The region where traction boundary condition is denoted by $\boldsymbol{\partial} \boldsymbol{\Omega}_{q}$ (see Fig.2). The traction on the lateral surface is given by $\boldsymbol{q}(s, Z)$. The traction on the top and the bottom surface is denoted by $\boldsymbol{q}^{+}(\boldsymbol{r}, 2 h)$ and $\boldsymbol{q}^{-}(\boldsymbol{r}, 0)$, respectively. Following [Anderson et al., 1999], we will assume that the strain energy density $\phi$ is only function of $\boldsymbol{F}, \boldsymbol{n}$ and $\boldsymbol{G}$. We will also neglect the deformational body force and the external orientational body force.

In the absence of any external body force, the potential energy, $E$, can be written as

$$
\begin{equation*}
E=\int_{V} \phi(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G}) d V-\int_{\partial V} \boldsymbol{t} \cdot \boldsymbol{u} d A \tag{8}
\end{equation*}
$$

where $\boldsymbol{t}$ is the traction on the boundaries and

$$
\begin{equation*}
u=x-X \tag{9}
\end{equation*}
$$

is the displacement of a material point. For our case, (8) reduces to

$$
\begin{equation*}
E(\boldsymbol{x}, \boldsymbol{n})=\int_{\Omega} \int_{0}^{2 h} \phi(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G}) d Z d \boldsymbol{r}-\int_{\Omega}\left\{\boldsymbol{q}^{-} \cdot \boldsymbol{u}(\boldsymbol{r}, 0)+\boldsymbol{q}^{+} \cdot \boldsymbol{u}(\boldsymbol{r}, 2 h)\right\} d \boldsymbol{r}-\int_{\partial \Omega_{q}} \int_{0}^{2 h} \boldsymbol{q}(s, Z) \cdot \boldsymbol{u}(s, Z) d Z d s \tag{10}
\end{equation*}
$$

The second term in the RHS is integration over the top and bottom surface while the last term in RHS is integration over the traction boundary. The expression in [Liu et al., 2020] on page 4 uses $\boldsymbol{x}$ instead of $\boldsymbol{u}$ which is not correct. However, this doesn't affect the final result, since variations in $\boldsymbol{u}$ is equal to variations in $\boldsymbol{x}$. At the equilibrium, the potential energy should be minimum. Therefore, we need to find the extremum of the functional, $E(\boldsymbol{x}, \boldsymbol{n})$, subject to the point-wise constraints (4)-(6). The modified functional now becomes,

$$
\begin{equation*}
\psi(\boldsymbol{x}, \boldsymbol{n})=E(\boldsymbol{x}, \boldsymbol{n})-\int_{\Omega} \int_{0}^{2 h} \lambda_{i}(\boldsymbol{X}) R_{i}(\boldsymbol{n}, \boldsymbol{F}, \boldsymbol{G}) d Z d \boldsymbol{r}, \quad i=1 . .3 \tag{11}
\end{equation*}
$$

where $\lambda_{i}(\boldsymbol{X})$ are Lagrangian multipliers. Consider the continuously differentiable functions $\boldsymbol{\eta}(\boldsymbol{X})$ and $\boldsymbol{\zeta}(\boldsymbol{X})$ s.t.

$$
\begin{align*}
& \boldsymbol{x}_{\epsilon}(\boldsymbol{X})=\boldsymbol{x}(\boldsymbol{X})+\epsilon \boldsymbol{\eta}(\boldsymbol{X}),  \tag{12}\\
& \boldsymbol{n}_{\epsilon}(\boldsymbol{X})=\boldsymbol{n}(\boldsymbol{X})+\epsilon \boldsymbol{\zeta}(\boldsymbol{X}), \tag{13}
\end{align*}
$$

for small $\epsilon . \boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are compactly supported only when displacement boundary conditions are specified on the whole boundary. From (9) and (12), we get

$$
\begin{equation*}
\boldsymbol{u}_{\epsilon}(\boldsymbol{X})=\boldsymbol{x}_{\epsilon}(\boldsymbol{X})-\boldsymbol{X}=\boldsymbol{x}(\boldsymbol{X})+\epsilon \boldsymbol{\eta}(\boldsymbol{X})-\boldsymbol{X}=\boldsymbol{u}(\boldsymbol{X})+\epsilon \boldsymbol{\eta}(\boldsymbol{X}) \tag{14}
\end{equation*}
$$

Calculating the first variation of $\psi$, we get

$$
\begin{equation*}
\left.\frac{d}{d \epsilon} \psi\left(\boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}\right)\right|_{\epsilon=0}=\left.\frac{d}{d \epsilon} E\left(\boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}\right)\right|_{\epsilon=0}-\left.\int_{\Omega} \int_{0}^{2 h} \frac{d}{d \epsilon}\left\{\lambda_{i}(\boldsymbol{X}) R_{i}\left(\boldsymbol{n}_{\epsilon}, \boldsymbol{F}_{\epsilon}, \boldsymbol{G}_{\epsilon}\right)\right\} d Z d \boldsymbol{r}\right|_{\epsilon=0} \quad i=1 . .3 \tag{15}
\end{equation*}
$$

The differentiation operator can be taken inside the integral because the integration is upon the fixed domain in the reference configuration. Using the Taylor series expansion for the energy density function, we obtain

$$
\begin{array}{r}
\phi\left(\boldsymbol{F}_{\epsilon}, \boldsymbol{n}_{\epsilon}, \boldsymbol{G}_{\epsilon}\right)=\phi\left(\nabla \boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}, \nabla \boldsymbol{n}_{\epsilon}\right)=\phi(\nabla \boldsymbol{x}+\epsilon \nabla \boldsymbol{\eta}, \boldsymbol{n}+\epsilon \boldsymbol{\zeta}, \nabla \boldsymbol{n}+\epsilon \nabla \boldsymbol{\zeta}) \\
=\phi(\nabla \boldsymbol{x}, \boldsymbol{n}, \nabla \boldsymbol{n})+\epsilon\left\{\frac{\partial}{\partial \nabla \boldsymbol{x}} \phi(\nabla \boldsymbol{x}, \boldsymbol{n}, \nabla \boldsymbol{n}): \nabla \boldsymbol{\eta}+\frac{\partial}{\partial \boldsymbol{n}} \phi(\nabla \boldsymbol{x}, \boldsymbol{n}, \nabla \boldsymbol{n}) \cdot \boldsymbol{\zeta}+\frac{\partial}{\partial \nabla \boldsymbol{n}} \phi(\nabla \boldsymbol{x}, \boldsymbol{n}, \nabla \boldsymbol{n}): \nabla \boldsymbol{\zeta}\right\}, \tag{16}
\end{array}
$$

where $\boldsymbol{A}: \boldsymbol{B}=A_{i j} B_{i j}$. Therefore,

$$
\begin{equation*}
\left.\frac{d \phi}{d \epsilon}\right|_{\epsilon=0}=\frac{\partial \phi}{\partial \boldsymbol{F}}: \nabla \boldsymbol{\eta}+\frac{\partial \phi}{\partial \boldsymbol{n}} \cdot \boldsymbol{\zeta}+\frac{\partial \phi}{\partial \boldsymbol{G}}: \nabla \boldsymbol{\zeta} . \tag{17}
\end{equation*}
$$

By differentiating (10) w.r.t $\epsilon$ and substituting (14) and (17), we obtain

$$
\begin{array}{r}
\left.\frac{d}{d \epsilon} E\left(\boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}\right)\right|_{\epsilon=0}=\int_{\Omega} \int_{0}^{2 h}\left\{\frac{\partial \phi}{\partial \boldsymbol{F}}: \nabla \boldsymbol{\eta}+\frac{\partial \phi}{\partial \boldsymbol{n}} \cdot \boldsymbol{\zeta}+\frac{\partial \phi}{\partial \boldsymbol{G}}: \nabla \boldsymbol{\zeta}\right\} d Z d \boldsymbol{r}-\int_{\Omega}\left\{\boldsymbol{q}^{-} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 0)+\boldsymbol{q}^{+} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 2 h)\right\} d \boldsymbol{r} \\
-\int_{\partial \Omega_{q}} \int_{0}^{2 h} \boldsymbol{q}(s, Z) \cdot \boldsymbol{\eta}(s, Z) d Z d s \tag{18}
\end{array}
$$

The above equation can be simplified using the following relation for an arbitrary tensor $\boldsymbol{P}$ and arbitrary vector $a$

$$
\begin{equation*}
\boldsymbol{P}: \nabla \boldsymbol{a}=P_{i j} \frac{\partial a_{i}}{\partial X_{j}}=\frac{\partial}{\partial X_{j}}\left(P_{i j} a_{i}\right)-\frac{\partial}{\partial X_{j}}\left(P_{i j}\right) a_{i}=\operatorname{Div}\left(\boldsymbol{T}^{T} \boldsymbol{a}\right)-\boldsymbol{a} \cdot \operatorname{Div}(\boldsymbol{T}) \tag{19}
\end{equation*}
$$

which when substituted in (18) becomes

$$
\begin{align*}
&\left.\frac{d}{d \epsilon} E\left(\boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}\right)\right|_{\epsilon=0}=-\int_{\Omega} \int_{0}^{2 h}\left\{\boldsymbol{\eta} \cdot \operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{F}}\right)+\boldsymbol{\zeta} \cdot\left[-\frac{\partial \phi}{\partial \boldsymbol{n}}+\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)\right]\right\} d Z d \boldsymbol{r}- \\
& \int_{\Omega} \int_{0}^{2 h}\left\{\operatorname{Div}\left({\frac{\partial \phi^{T}}{\partial \boldsymbol{F}}}^{T} \boldsymbol{\eta}\right)+\operatorname{Div}\left({\frac{\partial \phi^{T}}{\partial \boldsymbol{G}}}^{T} \boldsymbol{\zeta}\right)\right\} d Z d \boldsymbol{r}- \int_{\Omega}\left\{\boldsymbol{q}^{-} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 0)+\boldsymbol{q}^{+} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 2 h)\right\} d \boldsymbol{r} \\
&-\int_{\partial \Omega_{q}} \int_{0}^{2 h} \boldsymbol{q}(s, Z) \cdot \boldsymbol{\eta}(s, Z) d Z d s \tag{20}
\end{align*}
$$

Divergence theorem when applied to the geometry of the plate reduces to
$\int_{\Omega} \int_{0}^{2 h} \operatorname{Div}(\boldsymbol{a}) d Z d \boldsymbol{r}=\int_{\Omega} \boldsymbol{a}(\boldsymbol{r}, 2 h) \cdot \boldsymbol{k} d \boldsymbol{r}-\int_{\Omega} \boldsymbol{a}(\boldsymbol{r}, 0) \cdot \boldsymbol{k} d \boldsymbol{r}+\int_{\partial \Omega_{0}} \int_{0}^{2 h} \boldsymbol{a}(s, Z) \cdot \boldsymbol{N} d Z d s+\int_{\partial \Omega_{q}} \int_{0}^{2 h} \boldsymbol{a}(s, Z) \cdot \boldsymbol{N} d Z d s$,
for an arbitrary vector $\boldsymbol{a}$, where $\boldsymbol{k}$ is unit vector along the Z-axis and $\boldsymbol{N}$ is normal to the lateral boundaries of the plate. By substituting (21) in (20), we get

$$
\begin{gather*}
\left.\frac{d}{d \epsilon} E\left(\boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}\right)\right|_{\epsilon=0}=-\int_{\Omega} \int_{0}^{2 h}\left\{\boldsymbol{\eta} \cdot \operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{F}}\right)+\boldsymbol{\zeta} \cdot\left[-\frac{\partial \phi}{\partial \boldsymbol{n}}+\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)\right]\right\} d Z d \boldsymbol{r}-\int_{\Omega}{\frac{\partial \phi^{T}}{\partial \boldsymbol{F}}}^{\boldsymbol{\eta}(\boldsymbol{r}, 0) \cdot \boldsymbol{k} d \boldsymbol{r}} \\
+\int_{\Omega} \frac{\partial \phi^{T}}{\partial \boldsymbol{F}} \boldsymbol{\eta}(\boldsymbol{r}, 2 h) \cdot \boldsymbol{k} d \boldsymbol{r}-\int_{\partial \Omega_{q}} \int_{0}^{2 h} \frac{\partial \phi^{T}}{\partial \boldsymbol{F}} \boldsymbol{\eta}(s, Z) \cdot \boldsymbol{N} d Z d s-\int_{\Omega} \frac{\partial \phi^{T}}{\partial \boldsymbol{G}} \boldsymbol{\zeta}(\boldsymbol{r}, 0) \cdot \boldsymbol{k} d \boldsymbol{r}+\int_{\Omega} \frac{\partial \phi^{T}}{\partial \boldsymbol{G}} \boldsymbol{\zeta}(\boldsymbol{r}, 2 h) \cdot \boldsymbol{k} d \boldsymbol{r} \\
+\int_{\partial \Omega} \int_{0}^{2 h} \frac{\partial \phi^{T}}{\partial \boldsymbol{G}} \boldsymbol{\zeta}(s, Z) \cdot \boldsymbol{N} d Z d s-\int_{\Omega}\left\{\boldsymbol{q}^{-} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 0)+\boldsymbol{q}^{+} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 2 h)\right\} d \boldsymbol{r}-\int_{\partial \Omega_{q}} \int_{0}^{2 h} \boldsymbol{q}(s, Z) \cdot \boldsymbol{\eta}(s, Z) d Z d s \\
=-\int_{\Omega} \int_{0}^{2 h}\left\{\boldsymbol{\eta} \cdot \operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{F}}\right)+\boldsymbol{\zeta} \cdot\left[-\frac{\partial \phi}{\partial \boldsymbol{n}}+D i v\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)\right]\right\} d Z d \boldsymbol{r}-\int_{\Omega}\left\{\frac{\partial \phi}{\partial \boldsymbol{F}} \boldsymbol{k}+\boldsymbol{q}^{-}\right\} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 0) d \boldsymbol{r} \\
+\int_{\Omega}\left\{\frac{\partial \phi}{\partial \boldsymbol{F}} \boldsymbol{k}-\boldsymbol{q}^{+}\right\} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 2 h) d \boldsymbol{r}-\int_{\partial \Omega_{q}} \int_{0}^{2 h}\left\{\frac{\partial \phi}{\partial \boldsymbol{F}} \boldsymbol{N}-\boldsymbol{q}(s, Z)\right\} \cdot \boldsymbol{\eta}(s, Z) d Z d s-\int_{\Omega} \frac{\partial \phi}{\partial \boldsymbol{G}} \boldsymbol{k} \cdot \boldsymbol{\zeta}(\boldsymbol{r}, 0) d \boldsymbol{r} \\
+\int_{\Omega} \frac{\partial \phi}{\partial \boldsymbol{G}} \boldsymbol{k} \cdot \boldsymbol{\zeta}(\boldsymbol{r}, 2 h) d \boldsymbol{r}+\int_{\partial \Omega} \int_{0}^{2 h} \frac{\partial \phi}{\partial \boldsymbol{G}} \boldsymbol{N} \cdot \boldsymbol{\zeta}(s, Z) d Z d s \tag{22}
\end{gather*}
$$

Here, we have used the fact that on $\partial \Omega_{0}$ the displacement is specified and hence $\boldsymbol{\eta}(s, Z)$ vanishes there. Similarly,

$$
\begin{align*}
& \left.\int_{\Omega} \int_{0}^{2 h} \frac{d}{d \epsilon}\left\{\lambda_{i}(\boldsymbol{X}) R_{i}\left(\boldsymbol{n}_{\epsilon}, \boldsymbol{F}_{\epsilon}, \boldsymbol{G}_{\epsilon}\right)\right\} d Z d \boldsymbol{r}\right|_{\epsilon=0}=\left.\int_{\Omega} \int_{0}^{2 h} \frac{d}{d \epsilon}\left\{\lambda_{1}(\boldsymbol{X}) R_{1}\left(\boldsymbol{F}_{\epsilon}\right)+\lambda_{2}(\boldsymbol{X}) R_{2}\left(\boldsymbol{n}_{\epsilon}\right)+\lambda_{3}(\boldsymbol{X}) R_{3}\left(\boldsymbol{n}_{\epsilon}, \boldsymbol{G}_{\epsilon}\right)\right\} d Z d \boldsymbol{r}\right|_{\epsilon=0} \\
& =-\int_{\Omega} \int_{0}^{2 h}\left\{-\boldsymbol{\eta} \cdot \operatorname{Div}\left(\lambda_{1} \frac{\partial R_{1}}{\partial \boldsymbol{F}}\right)+\boldsymbol{\zeta} \cdot\left[\lambda_{2} \frac{\partial R_{2}}{\partial \boldsymbol{n}}+\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{n}}-\operatorname{Div}\left(\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}}\right)\right]\right\} d Z d \boldsymbol{r} \\
& -\int_{\Omega} \int_{0}^{2 h}\left\{\operatorname{Div}\left(\lambda_{1} \frac{\partial R_{1}{ }^{T}}{\partial \boldsymbol{F}} \boldsymbol{\eta}\right)+\operatorname{Div}\left(\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}} \boldsymbol{\zeta}\right)\right\} d Z d \boldsymbol{r} \\
& =-\int_{\Omega} \int_{0}^{2 h}\left\{-\boldsymbol{\eta} \cdot \operatorname{Div}\left(\lambda_{1} \frac{\partial R_{1}}{\partial \boldsymbol{F}}\right)+\boldsymbol{\zeta} \cdot\left[\lambda_{2} \frac{\partial R_{2}}{\partial \boldsymbol{n}}+\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{n}}-\operatorname{Div}\left(\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}}\right)\right]\right\} d Z d \boldsymbol{r}-\int_{\Omega} \lambda_{1} \frac{\partial R_{1}}{\partial \boldsymbol{F}} \boldsymbol{k} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 0) d \boldsymbol{r} \\
& +\int_{\Omega} \lambda_{1} \frac{\partial R_{1}}{\partial \boldsymbol{F}} \boldsymbol{k} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 2 h) d \boldsymbol{r}-\int_{\partial \Omega_{q}} \int_{0}^{2 h} \lambda_{1} \frac{\partial R_{1}}{\partial \boldsymbol{F}} \boldsymbol{N} \cdot \boldsymbol{\eta}(s, Z) d Z d s-\int_{\Omega} \lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}} \boldsymbol{k} \cdot \boldsymbol{\zeta}(\boldsymbol{r}, 0) d \boldsymbol{r} \\
& +\int_{\Omega} \lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}} \boldsymbol{k} \cdot \boldsymbol{\zeta}(\boldsymbol{r}, 2 h) d \boldsymbol{r}+\int_{\partial \Omega} \int_{0}^{2 h} \lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}} \boldsymbol{N} \cdot \boldsymbol{\zeta}(s, Z) d Z d s \tag{23}
\end{align*}
$$

Combining, (22) and (23), gives

$$
\begin{align*}
\left.\frac{d}{d \epsilon} \psi\left(\boldsymbol{x}_{\epsilon}, \boldsymbol{n}_{\epsilon}\right)\right|_{\epsilon=0} & =-\int_{\Omega} \int_{0}^{2 h} \operatorname{Div}(\boldsymbol{S}) \cdot \boldsymbol{\eta} d Z d \boldsymbol{r}-\int_{\Omega}\left\{\boldsymbol{S} \boldsymbol{k}+\boldsymbol{q}^{-}\right\} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 0) d \boldsymbol{r}+\int_{\Omega}\left\{\boldsymbol{S} \boldsymbol{k}-\boldsymbol{q}^{+}\right\} \cdot \boldsymbol{\eta}(\boldsymbol{r}, 2 h) d \boldsymbol{r} \\
& -\int_{\partial \Omega_{q}} \int_{0}^{2 h}\{\boldsymbol{S} \boldsymbol{k}-\boldsymbol{q}(s, Z)\} \cdot \boldsymbol{\eta}(s, Z) d Z d s-\int_{\Omega} \int_{0}^{2 h}\{D i v(\boldsymbol{T})+\boldsymbol{\pi}\} \cdot \boldsymbol{\zeta} d Z d \boldsymbol{r} \\
& -\int_{\Omega} \boldsymbol{T} \boldsymbol{k} \cdot \boldsymbol{\zeta}(\boldsymbol{r}, 0) d \boldsymbol{r}+\int_{\Omega} \boldsymbol{T} \boldsymbol{k} \cdot \boldsymbol{\zeta}(\boldsymbol{r}, 2 h) d \boldsymbol{r}+\int_{\partial \Omega} \int_{0}^{2 h} \boldsymbol{T} \boldsymbol{N} \cdot \boldsymbol{\zeta}(s, Z) d Z d s \tag{24}
\end{align*}
$$

where

$$
\begin{equation*}
\boldsymbol{S}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G})=\frac{\partial \phi}{\partial \boldsymbol{F}}-\lambda_{1} \frac{\partial R_{1}}{\partial \boldsymbol{F}} \tag{25}
\end{equation*}
$$

is the nominal stress tensor,

$$
\begin{equation*}
\boldsymbol{T}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G})=\frac{\partial \phi}{\partial \boldsymbol{G}}-\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{G}} \tag{26}
\end{equation*}
$$

is the orientational stress tensor and

$$
\begin{equation*}
\boldsymbol{\pi}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G})=-\frac{\partial \phi}{\partial \boldsymbol{n}}+\lambda_{2} \frac{\partial R_{2}}{\partial \boldsymbol{n}}+\lambda_{3} \frac{\partial R_{3}}{\partial \boldsymbol{n}} \tag{27}
\end{equation*}
$$

is the internal orientational body force. At the extremum, the first variation should be zero and also since $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are arbitrary, from localization principle, we obtain following equilibrium conditions,

$$
\begin{align*}
\operatorname{Div}(\boldsymbol{S}) & =0, \\
\operatorname{Div}(\boldsymbol{T})+\boldsymbol{\pi} & =0 \tag{28}
\end{align*}
$$

and following natural boundary conditions,

$$
\begin{array}{cc}
\left.\boldsymbol{S k}\right|_{z=0}=-\boldsymbol{q}^{-}(\boldsymbol{r}),\left.\boldsymbol{S k}\right|_{z=2 h}=\boldsymbol{q}^{+}(\boldsymbol{r}) & \text { on the top and bottom boundaries } \Omega, \\
\boldsymbol{S} \boldsymbol{N}=\boldsymbol{q}(s, Z), & \text { on the lateral traction boundaries } \partial \Omega_{q} \times[0,2 h], \\
\left.\boldsymbol{T} \boldsymbol{k}\right|_{z=0,2 h}=0 & \text { on the top and bottom boundaries } \Omega, \\
\boldsymbol{T} \boldsymbol{N}=0, & \text { on the lateral boundaries } \partial \Omega \times[0,2 h] .
\end{array}
$$

Note that the boundary conditions,(29), are different than that of [Liu et al., 2020] 2.11. This is because of the definition of the divergence. We have used the definition of divergence of a tensor $\boldsymbol{P}$ as

$$
\operatorname{Div}(\boldsymbol{P}) \cdot \boldsymbol{a}=\operatorname{Div}\left(\boldsymbol{P}^{T} \boldsymbol{a}\right)
$$

where $\boldsymbol{a}$ is a constant vector while [Liu et al., 2020] have used the definition

$$
\operatorname{Div}(\boldsymbol{P}) \cdot \boldsymbol{a}=\operatorname{Div}(\boldsymbol{P a})
$$

$\boldsymbol{S}$ is the stress tensor and hence the lagrange multiplier $\lambda_{1}$ is also called the pressure. The $(28)_{2}$ is the evolution equation of the director. The Div operator indicates that the divergence is taken in the reference configuration. The expressions (25)-(27) involves $R_{1}, R_{2}$ and $R_{3}$ which can be further simplified using the following result,

$$
\begin{aligned}
& \nabla_{\boldsymbol{F}} \operatorname{det}(\boldsymbol{F}): d \boldsymbol{F}=d(\operatorname{det} \boldsymbol{F})=d\left(\frac{\boldsymbol{F} \boldsymbol{a} \cdot(\boldsymbol{F b} \times \boldsymbol{F} \boldsymbol{c})}{\boldsymbol{a} .(\boldsymbol{b} \times \boldsymbol{c})}\right), \quad \quad\left(\nabla_{\boldsymbol{F}}=\frac{\partial}{\partial \boldsymbol{F}}\right) \\
& =\left(\frac{d F a .(F b \times F c)}{a .(b \times c)}+\frac{F a \cdot(d F b \times F c)}{a .(b \times c)}+\frac{F a \cdot(F b \times d F c)}{a \cdot(b \times c)}\right), \\
& =\left(\frac{d F F^{-1} F a .(F b \times F c)}{a .(b \times c)}+\frac{F a .\left(d F F^{-1} F b \times F c\right)}{a .(b \times c)}+\frac{F a .\left(F b \times d F F^{-1} F c\right)}{a .(b \times c)}\right), \\
& =\left(\frac{d F F^{-1} F a .(F b \times F c)}{F a .(F b \times F c)}+\frac{F a .\left(d F F^{-1} F b \times F c\right)}{F a .(F b \times F c)}+\frac{F a .\left(F b \times d F F^{-1} F c\right)}{F a .(F b \times F c)}\right)\left(\frac{F a .(F b \times F c)}{a .(b \times c)}\right), \\
& =\operatorname{tr}\left(d \boldsymbol{F} \boldsymbol{F}^{-1}\right) \operatorname{det}(\boldsymbol{F})=\operatorname{det}(F) \boldsymbol{F}^{-T}: d \boldsymbol{F}
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\nabla_{\boldsymbol{F}} \operatorname{det}(\boldsymbol{F})=\frac{\partial}{\partial \boldsymbol{F}}=\operatorname{det}(F) \boldsymbol{F}^{-T} \tag{30}
\end{equation*}
$$

Now, from (4)-(6) and (30), we obtain

$$
\begin{align*}
& \frac{\partial R_{1}}{\partial \boldsymbol{F}}=\operatorname{det}(\boldsymbol{F}) \boldsymbol{F}^{-T}=\boldsymbol{F}^{-T} \\
& \frac{\partial R_{2}}{\partial \boldsymbol{n}}=\frac{\partial}{\partial \boldsymbol{n}} \boldsymbol{n} . \boldsymbol{n}=2 \boldsymbol{n}  \tag{31}\\
& \frac{\partial R_{3}}{\partial \boldsymbol{n}}=\frac{\partial}{\partial \boldsymbol{n}} \boldsymbol{n} . \boldsymbol{G} \boldsymbol{a}=\boldsymbol{G} \boldsymbol{a} \\
& \frac{\partial R_{3}}{\partial \boldsymbol{G}}=\frac{\partial}{\partial \boldsymbol{G}} \boldsymbol{n} . \boldsymbol{G} \boldsymbol{a}=\frac{\partial}{\partial \boldsymbol{G}} \boldsymbol{G}: \boldsymbol{n} \otimes \boldsymbol{a}=\boldsymbol{n} \otimes \boldsymbol{a}
\end{align*}
$$

Substituting (31) in (25)-(27), we obtain

$$
\begin{align*}
\boldsymbol{S}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G}) & =\frac{\partial \phi}{\partial \boldsymbol{F}}-\lambda_{1} \boldsymbol{F}^{-T} \\
\boldsymbol{\pi}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G}) & =-\frac{\partial \phi}{\partial \boldsymbol{n}}+\lambda_{2} \boldsymbol{n}+\lambda_{3} \boldsymbol{G} \boldsymbol{a} \\
\boldsymbol{T}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G}) & =\frac{\partial \phi}{\partial \boldsymbol{G}}-\lambda_{3} \boldsymbol{n} \otimes \boldsymbol{a} \tag{32}
\end{align*}
$$

Again, (32) varies from the expressions in 2.16 of [Liu et al., 2020] but is similar to 4.25 of [Anderson et al., 1999]. The equations (4)-(6) and (28) is a set of five 3D governing equations for five variables $\boldsymbol{x}, \boldsymbol{n}, \lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ with boundary conditions (7) and (29). The multipliers $\lambda_{2}$ and $\lambda_{3}$ are of negligible importance ([Anderson et al., 1999]). We can eliminate them by operating $(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})$ on $(28)_{2},(29)_{3}$ and $(29)_{4}$. The $(28)_{1}$ becomes

$$
\begin{align*}
(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})(\operatorname{Div}(\boldsymbol{T})+\boldsymbol{\pi}) & =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}-\lambda_{3} \boldsymbol{n} \otimes \boldsymbol{a}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}+\lambda_{2} \boldsymbol{n}+\lambda_{3} \boldsymbol{G} \boldsymbol{a}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}-\operatorname{Div}\left(\lambda_{3} \boldsymbol{n} \otimes \boldsymbol{a}\right)+\lambda_{2} \boldsymbol{n}+\lambda_{3} \boldsymbol{G} \boldsymbol{a}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}-(\boldsymbol{n} \otimes \boldsymbol{a}) \nabla \lambda_{3}-\lambda_{3} \nabla(\boldsymbol{n}) \boldsymbol{a}+\lambda_{2} \boldsymbol{n}+\lambda_{3} \boldsymbol{G} \boldsymbol{a}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}+\left(\lambda_{2}-\boldsymbol{a} \cdot \nabla \lambda_{3}\right) \boldsymbol{n}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}\right)+(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\left(\lambda_{2}-\boldsymbol{a} \cdot \nabla \lambda_{3}\right) \boldsymbol{n}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}\right)+(\boldsymbol{n}-\boldsymbol{n})\left(\lambda_{2}-\boldsymbol{a} \cdot \nabla \lambda_{3}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})\left(\operatorname{Div}\left(\frac{\partial \phi}{\partial \boldsymbol{G}}\right)-\frac{\partial \phi}{\partial \boldsymbol{n}}\right) \\
& =(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})(\operatorname{Div} \tilde{\boldsymbol{T}}+\tilde{\boldsymbol{\pi}}) \tag{33}
\end{align*}
$$

where

$$
\begin{equation*}
\tilde{\boldsymbol{T}}=\partial \phi / \partial \boldsymbol{G} \quad \text { and } \quad \tilde{\boldsymbol{\pi}}=-\partial \phi / \partial \boldsymbol{n} \tag{34}
\end{equation*}
$$

Similarly, $(29)_{3}$ and $(29)_{4}$ becomes

$$
\begin{align*}
\left.(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n}) \tilde{\boldsymbol{T}} \boldsymbol{k}\right|_{z=0,2 h} & =0 \quad \text { on the top and bottom boundaries } \Omega \\
(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n}) \tilde{\boldsymbol{T}} \boldsymbol{N} & =0, \quad \text { on the lateral boundaries } \partial \Omega \times[0,2 h] . \tag{35}
\end{align*}
$$

The tensor $(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})$ can be written as

$$
\begin{aligned}
(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n}) & =\boldsymbol{n} \otimes \boldsymbol{n}+\boldsymbol{m} \otimes \boldsymbol{m}+\boldsymbol{l} \otimes \boldsymbol{l}-\boldsymbol{n} \otimes \boldsymbol{n} \\
& =\boldsymbol{m} \otimes \boldsymbol{m}+\boldsymbol{l} \otimes \boldsymbol{l}
\end{aligned}
$$

where $\boldsymbol{l}, \boldsymbol{m}$ and $\boldsymbol{n}$ forms an orthonormal basis. From the above result, we conclude that the rank of $(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})$ is two and hence (33) gives only two independent equations. [Anderson et al., 1999] proposed the following form of the energy density function

$$
\begin{equation*}
\phi=\frac{\mu_{f}}{2}\left(\operatorname{tr}\left(\boldsymbol{L}^{-1} \boldsymbol{F} \boldsymbol{L}_{0} \boldsymbol{F}^{T}\right)-3\right)+\frac{k(s-1)^{2}}{2 s} \operatorname{tr}\left(\boldsymbol{F}^{T} \boldsymbol{G} \boldsymbol{G}^{T} \boldsymbol{F}\right) \tag{36}
\end{equation*}
$$

where $\mu$ is the shear modulus and $k$ is the Frank constant. $\boldsymbol{L}$ and $\boldsymbol{L}_{0}$ are step-length tensors given by

$$
\begin{array}{r}
\boldsymbol{L}_{0}=l_{\perp}^{0} \boldsymbol{I}+\left(l_{\|}^{0}-l_{\perp}^{0}\right) \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0} \\
\boldsymbol{L}=l_{\perp} \boldsymbol{I}+\left(l_{\|}-l_{\perp}\right) \boldsymbol{n} \otimes \boldsymbol{n} . \tag{37}
\end{array}
$$

$l_{\perp}^{0}$ and $l_{\|}^{0}$ are positive constants which represent the step lengths perpendicular and parallel to the director in the reference configuration. Similarly, $l_{\perp}$ and $l_{\|}$represents the step lengths perpendicular and parallel to the director in the current configuration. $s$ is the step length anisotropy which is the ratio of $l_{\|}$and $l_{\perp}$. We will assume that the step lengths remains same in the reference and the current configuration and hence

$$
l_{\|}=l_{\|}^{0}, \quad l_{\perp}=l_{\perp}^{0}, \quad s=\frac{l_{\|}}{l_{\perp}}=\frac{l_{\|}^{0}}{l_{\perp}^{0}}>0
$$

$s>1$ means that the molecule is prolate and $s<1$ means oblate. For $s=1$, the molecule is spherical. The last term of (36) comes from the classical Oseen-Zocher-Frank energy density function for nematic liquid crystals and the first term is is the comes from classical neo-Hookean model for hyper-elastic materials. When $s=1$, $\boldsymbol{L}=\boldsymbol{L}_{0}$ becomes identity and the model reduces to the standard neo-Hookean model for isotropic material. The inverse of $\boldsymbol{L}$ is given by

$$
\begin{equation*}
\boldsymbol{L}^{-1}=l_{\perp}^{-1} \boldsymbol{I}+\left(l_{\|}^{-1}-l_{\perp}^{-1}\right) \boldsymbol{n} \otimes \boldsymbol{n} \tag{38}
\end{equation*}
$$

using which we obtain,

$$
\begin{align*}
\operatorname{tr}\left(\boldsymbol{L}^{-1} \boldsymbol{F} \boldsymbol{L}_{0} \boldsymbol{F}^{T}\right) & =\boldsymbol{L}^{-1} \boldsymbol{F}: \boldsymbol{F} \boldsymbol{L}_{0}^{T} \\
& =\left(\frac{1}{l_{\perp}} \boldsymbol{F}+\left(\frac{1}{l_{\|}}-\frac{1}{l_{\perp}}\right) \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}\right):\left(l_{\perp}^{0} \boldsymbol{F}+\left(l_{\|}^{0}-l_{\perp}^{0}\right) \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0}\right) \\
& =\boldsymbol{F}: \boldsymbol{F}-\frac{s-1}{s} \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}: \boldsymbol{F}+(s-1) \boldsymbol{F}: \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0}-\frac{(s-1)^{2}}{s} \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}: \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0} \\
& =\boldsymbol{F}: \boldsymbol{F}-\frac{s-1}{s} \boldsymbol{F}^{T} \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}: \boldsymbol{I}+(s-1) \boldsymbol{I}: \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{F} \boldsymbol{n}_{0}-\frac{(s-1)^{2}}{s} \boldsymbol{F}^{T} \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}: \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0} \\
& =\operatorname{tr}\left(\boldsymbol{F} \boldsymbol{F}^{T}\right)-\frac{s-1}{s} \boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{F}^{T} \boldsymbol{n}+(s-1) \boldsymbol{F} \boldsymbol{n}_{0} \cdot \boldsymbol{F} \boldsymbol{n}_{0}-\frac{(s-1)^{2}}{s}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right)^{2} \tag{39}
\end{align*}
$$

Substituting (39) in (36), we find the final form of the strain energy function as
$\phi=\frac{\mu}{2}\left(\operatorname{tr}\left(\boldsymbol{F} \boldsymbol{F}^{T}\right)-\frac{s-1}{s} \boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{F}^{T} \boldsymbol{n}+(s-1) \boldsymbol{F} \boldsymbol{n}_{0} \cdot \boldsymbol{F} \boldsymbol{n}_{0}-\frac{(s-1)^{2}}{s}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right)^{2}-3\right)+\frac{k(s-1)^{2}}{2 s} \operatorname{tr}\left(\boldsymbol{F}^{T} \boldsymbol{G} \boldsymbol{G}^{T} \boldsymbol{F}\right)$.
Using the above form of energy density, we can calculate the exact form of deformational tensor $\boldsymbol{S}$ from (32) ${ }_{1}$, orientational tensor $\tilde{\boldsymbol{T}}$ and internal orientational body force $\tilde{\boldsymbol{\pi}}$ from (34) in terms of $\boldsymbol{F}, \boldsymbol{n}$ and $\boldsymbol{G}$. To get the exact form, we need to use the following results,

$$
\begin{align*}
\frac{\partial}{\partial \boldsymbol{F}} \operatorname{tr}\left(\boldsymbol{F} \boldsymbol{F}^{T}\right) & =\frac{\partial}{\partial \boldsymbol{F}}(\boldsymbol{F}: \boldsymbol{F})=2 \boldsymbol{F} \\
\frac{\partial}{\partial \boldsymbol{F}} \boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{F}^{T} \boldsymbol{n} & =\frac{\partial}{\partial F_{m n}}\left(F_{j i} n_{j} F_{k i} n_{k}\right) \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2\left(F_{k n} n_{k} n_{m}\right) \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2 \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}  \tag{41}\\
\frac{\partial}{\partial \boldsymbol{F}} \boldsymbol{F} \boldsymbol{n}_{0} \cdot \boldsymbol{F} \boldsymbol{n}_{0} & =\frac{\partial}{\partial F_{m n}}\left(F_{i j} n_{j} F_{i k} n_{k}\right) \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2\left(F_{m k} n_{k} n_{n}\right) \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2 \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0}
\end{align*}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \boldsymbol{F}}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right)^{2} & =2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \frac{\partial}{\partial \boldsymbol{F}}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right)=2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \frac{\partial}{\partial \boldsymbol{F}}\left(\boldsymbol{F}^{T}: \boldsymbol{n}_{0} \otimes \boldsymbol{n}\right)=2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \frac{\partial}{\partial \boldsymbol{F}}\left(\boldsymbol{F}: \boldsymbol{n} \otimes \boldsymbol{n}_{0}\right) \\
& =2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \boldsymbol{n} \otimes \boldsymbol{n}_{0} \\
\frac{\partial}{\partial \boldsymbol{F}} \operatorname{tr}\left(\boldsymbol{F}^{T} \boldsymbol{G} \boldsymbol{G}^{T} \boldsymbol{F}\right) & =\frac{\partial}{\partial \boldsymbol{F}}\left(\boldsymbol{F}^{T} \boldsymbol{G}^{T} \boldsymbol{G}^{T}: \boldsymbol{F}^{T}\right)=\frac{\partial}{\partial F_{m n}}\left(F_{j i} G_{j k} G_{l k} F_{l i}\right) \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2 G_{m k} G_{l k} F_{l n} \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2 \boldsymbol{G} \boldsymbol{G}^{T} \boldsymbol{F} \\
\frac{\partial}{\partial \boldsymbol{G}}\left(t r\left(\boldsymbol{F}^{T} \boldsymbol{G} \boldsymbol{G}^{T} \boldsymbol{F}\right)\right) & =\frac{\partial}{\partial \boldsymbol{G}}\left(\boldsymbol{F}^{T} \boldsymbol{G}: \boldsymbol{F}^{T} \boldsymbol{G}\right)=\frac{\partial}{\partial G_{m n}}\left(F_{j i} G_{j k} G_{l k} F_{l i}\right) \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2 F_{m i} G_{l n} F_{l i} \boldsymbol{e}_{m} \otimes \boldsymbol{e}_{n}=2 \boldsymbol{F} \boldsymbol{F}^{T} \boldsymbol{G} \\
\frac{\partial}{\partial \boldsymbol{n}} \boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{F}^{T} \boldsymbol{n} & =2 \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{m}}}\left(F_{j i} n_{j} F_{k i} n_{k}\right) \boldsymbol{e}_{m}=2\left(F_{m i} F_{k i} n_{k}\right) \boldsymbol{e}_{m}=2 \boldsymbol{F} \boldsymbol{F}^{T} \boldsymbol{n} \\
\frac{\partial}{\partial \boldsymbol{n}}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right)^{2} & =2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \frac{\partial}{\partial \boldsymbol{n}}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right)=2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \frac{\partial}{\partial \boldsymbol{n}}\left(\boldsymbol{n} \cdot \boldsymbol{F} \boldsymbol{n}_{0}\right)=2\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \boldsymbol{F} \boldsymbol{n}_{0}=\left(\boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{F} \boldsymbol{n}_{0}\right) \boldsymbol{n}
\end{aligned}
$$

Substituting the above results and (40) in $(32)_{1}$ and (34), we get

$$
\begin{align*}
& \boldsymbol{S}=\mu\left(\boldsymbol{F}-\frac{s-1}{s} \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}+(s-1) \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0}-\frac{(s-1)^{2}}{s}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \boldsymbol{n} \otimes \boldsymbol{n}_{0}\right)+\frac{k(s-1)^{2}}{s} \boldsymbol{G}^{T} \boldsymbol{F}-\lambda_{1} \boldsymbol{F}^{-T}  \tag{42}\\
& \tilde{\boldsymbol{T}}=\frac{k(s-1)^{2}}{2 s} \boldsymbol{F} \boldsymbol{F}^{T} \boldsymbol{G}  \tag{43}\\
& \tilde{\boldsymbol{\pi}}=-\frac{s-1}{s}\left(\boldsymbol{F} \boldsymbol{F}^{T}+(s-1) \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{F} \boldsymbol{n}_{0}\right) \boldsymbol{n} \tag{44}
\end{align*}
$$

We now have the complete set of reduced 3D governing equations

$$
\begin{array}{r}
\operatorname{Div}(\boldsymbol{S})=0 \\
(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n})(\operatorname{Div} \tilde{\boldsymbol{T}}+\tilde{\boldsymbol{\pi}})=0 \tag{45}
\end{array}
$$

where $\boldsymbol{S}, \tilde{\boldsymbol{T}}$ and $\tilde{\boldsymbol{\pi}}$ are defined in (42)-(44) and boundary conditions given by

$$
\begin{array}{cc}
\left.\boldsymbol{S} \boldsymbol{k}\right|_{z=0}=-\boldsymbol{q}^{-}(\boldsymbol{r}),\left.\boldsymbol{S k}\right|_{z=2 h}=\boldsymbol{q}^{+}(\boldsymbol{r}) & \text { on the top and bottom boundaries } \Omega, \\
\boldsymbol{S} \boldsymbol{N}=\boldsymbol{q}(s, Z), & \text { on the lateral traction boundaries } \partial \Omega_{q} \times[0,2 h],  \tag{46}\\
\left.(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n}) \tilde{\boldsymbol{T}} \boldsymbol{k}\right|_{z=0,2 h}=0 & \text { on the top and bottom boundaries } \Omega, \\
(\boldsymbol{I}-\boldsymbol{n} \otimes \boldsymbol{n}) \tilde{\boldsymbol{T}} \boldsymbol{N}=0, & \text { on the lateral boundaries } \partial \Omega \times[0,2 h],
\end{array}
$$

for the variables $\boldsymbol{x}, \boldsymbol{n}$ and $\lambda_{1}$ subject to the constraints

$$
\begin{equation*}
\operatorname{det}(\boldsymbol{F})=1 \quad \text { and } \quad \boldsymbol{n} \cdot \boldsymbol{n}=1 \tag{47}
\end{equation*}
$$

We have 7 equations ( 3 from $(45)_{1}, 2$ from $(45)_{2}$ and 2 from the constraints (47)). The total number of variables is also 7 ( 3 from $\boldsymbol{x}, 3$ from $\boldsymbol{n}$ and $\lambda_{1}$ ). Hence our system of equations is closed.

## 2D vector plate equations

We can do a series expansion to get the 2D plate theory by assuming that the thickness of the plate $2 h$ is much smaller than the other two dimensions[Liu et al., 2020]. The series expansion about the bottom surface for an arbitrary tensor/vector $\boldsymbol{P}(\boldsymbol{X})$ upto the fourth order is given as

$$
\begin{equation*}
\boldsymbol{P}(\boldsymbol{X})=\boldsymbol{P}^{(0)}(\boldsymbol{r})+Z \boldsymbol{P}^{(1)}(\boldsymbol{r})+\frac{1}{2} Z^{2} \boldsymbol{P}^{(2)}(\boldsymbol{r})+\frac{1}{6} Z^{3} \boldsymbol{P}^{(3)}(\boldsymbol{r})+\frac{1}{24} Z^{4} \boldsymbol{P}^{(4)}(\boldsymbol{r})+\mathcal{O}\left(h^{5}\right) \tag{48}
\end{equation*}
$$

where

$$
\boldsymbol{P}^{(m)}=\frac{\partial^{m}}{\partial Z^{m}} \boldsymbol{P}(X)
$$

The quantities $\boldsymbol{x}(\boldsymbol{X})$ and $\boldsymbol{F}(\boldsymbol{X})$ can be expanded in the Taylor series by substituting $\boldsymbol{P}(\boldsymbol{X})=\boldsymbol{x}(\boldsymbol{X})$ and $\boldsymbol{P}(\boldsymbol{X})=\boldsymbol{F}(\boldsymbol{X})$ in (48), respectively. The series expansion of $\boldsymbol{x}(\boldsymbol{X})$ when substituted in (2) gives,

$$
\begin{aligned}
\boldsymbol{F}= & \bar{\nabla} \boldsymbol{x}(\boldsymbol{r}, Z)+\frac{\partial \boldsymbol{x}(\boldsymbol{r}, Z)}{\partial Z} \otimes \boldsymbol{k}, \\
= & \bar{\nabla}\left(\boldsymbol{P}^{(0)}(\boldsymbol{r})\right)+Z \bar{\nabla}\left(\boldsymbol{P}^{(1)}(\boldsymbol{r})\right)+\frac{1}{2} Z^{2} \bar{\nabla}\left(\boldsymbol{P}^{(2)}(\boldsymbol{r})\right)+\frac{1}{6} Z^{3} \bar{\nabla}\left(\boldsymbol{P}^{(3)}(\boldsymbol{r})\right)+\frac{1}{24} Z^{4} \bar{\nabla}\left(\boldsymbol{P}^{(4)}(\boldsymbol{r})\right) \\
& +\boldsymbol{P}^{(1)}(\boldsymbol{r}) \otimes \boldsymbol{k}+Z \boldsymbol{P}^{(2)}(\boldsymbol{r}) \otimes \boldsymbol{k}+\frac{1}{2} Z^{2} \boldsymbol{P}^{(3)}(\boldsymbol{r}) \otimes \boldsymbol{k}+\frac{1}{6} Z^{3} \boldsymbol{P}^{(4)}(\boldsymbol{r}) \otimes \boldsymbol{k}+\mathcal{O}\left(Z^{4}\right) \\
= & \left\{\bar{\nabla}\left(\boldsymbol{P}^{(0)}(\boldsymbol{r})\right)+\boldsymbol{P}^{(1)}(\boldsymbol{r}) \otimes \boldsymbol{k}\right\} \\
& +Z\left\{\bar{\nabla}\left(\boldsymbol{P}^{(1)}(\boldsymbol{r})\right)+\boldsymbol{P}^{(2)}(\boldsymbol{r}) \otimes \boldsymbol{k}\right\}+\frac{1}{2} Z^{2}\left\{\bar{\nabla}\left(\boldsymbol{P}^{(2)}(\boldsymbol{r})\right)+\boldsymbol{P}^{(3)}(\boldsymbol{r}) \otimes \boldsymbol{k}\right\} \\
& +\frac{1}{6} Z^{3}\left\{3 \bar{\nabla}\left(\boldsymbol{P}^{(3)}(\boldsymbol{r})\right)+\boldsymbol{P}^{(4)}(\boldsymbol{r}) \otimes \boldsymbol{k}\right\}+\mathcal{O}\left(Z^{4}\right)
\end{aligned}
$$

Comparing with the Taylor series expansion of $\boldsymbol{F}$, we get the following relations,

$$
\begin{equation*}
\boldsymbol{F}^{(i)}=\bar{\nabla} \boldsymbol{x}^{(i)}+\boldsymbol{x}^{(i+1)} \boldsymbol{k}, \quad i=1,2,3 \ldots \tag{49}
\end{equation*}
$$

Similar relation can be found between orientation gradient tensor $\boldsymbol{G}$ and the director $\boldsymbol{n}$ by following the same procedure i.e.

$$
\begin{equation*}
\boldsymbol{G}^{(i)}=\bar{\nabla} \boldsymbol{n}^{(i)}+\boldsymbol{n}^{(i+1)} \boldsymbol{k}, \quad i=1,2,3 \ldots \tag{50}
\end{equation*}
$$

The nominal stress tensor $\boldsymbol{S}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G})$ is a multivariate function of $\boldsymbol{F}, \boldsymbol{n}$ and $\boldsymbol{G}$. Using the Taylor series expansion for multivariate function, we can expand $\boldsymbol{S}$ in terms of $\left(\boldsymbol{F}^{(0)}, \boldsymbol{n}^{(0)}, \boldsymbol{G}^{(0)}\right)$. Substituting the series expansion of $\boldsymbol{F}, \boldsymbol{n}$ and $\boldsymbol{G}$ in the multivariate expansion of $\boldsymbol{S}$, we obtain a complex but linear relation between $\boldsymbol{S}^{(i)}$ and ( $\boldsymbol{x}^{i+1}, \boldsymbol{n}^{(i+1)}, \lambda_{1}^{(i)}$ ), similar to (49) (see [Liu et al., 2020] for details). Similar procedure can also be followed for the orientational stress tensor $\tilde{\boldsymbol{T}}$ and the internal orientational body force $\tilde{\boldsymbol{\pi}}$.

The governing equation $(45)_{1}$ when used with expanded form of $\boldsymbol{S}$ becomes,

$$
\begin{align*}
\operatorname{Div}(\boldsymbol{S})= & \operatorname{Div}\left(\boldsymbol{S}^{(0)}(\boldsymbol{r})+Z \boldsymbol{S}^{(1)}(\boldsymbol{r})+\frac{1}{2} Z^{2} \boldsymbol{S}^{(2)}(\boldsymbol{r})+\frac{1}{6} Z^{3} \boldsymbol{S}^{(3)}(\boldsymbol{r})+\frac{1}{24} Z^{4} \boldsymbol{S}^{(4)}(\boldsymbol{r})\right) \\
= & \left.\bar{\nabla} \cdot\left(\boldsymbol{S}^{(0)}(\boldsymbol{r})\right)+\boldsymbol{S}^{(1)}(\boldsymbol{r}) \boldsymbol{k}+Z\left\{\bar{\nabla} \cdot\left(\boldsymbol{S}^{(1)}(\boldsymbol{r})\right)+\boldsymbol{S}^{(2)} \boldsymbol{r}\right) \boldsymbol{k}\right\}+\frac{1}{2} Z^{2}\left\{\bar{\nabla} \cdot\left(\boldsymbol{S}^{(2)}(\boldsymbol{r})\right)+\boldsymbol{S}^{(3)}(\boldsymbol{r}) \boldsymbol{k}\right\} \\
& +\frac{1}{6} Z^{3}\left\{\bar{\nabla} \cdot\left(\boldsymbol{S}^{(3)}(\boldsymbol{r})\right)+\boldsymbol{S}^{(4)}(\boldsymbol{r}) \boldsymbol{k}\right\}+\mathcal{O}\left(Z^{4}\right) \tag{51}
\end{align*}
$$

from which we get the relations

$$
\begin{equation*}
\bar{\nabla} \cdot \boldsymbol{S}^{(i)}+\boldsymbol{S}^{(i+1)} \boldsymbol{k}=0 \tag{52}
\end{equation*}
$$

Similar relations can be found by substituting expansion of $\boldsymbol{G}$ and $\boldsymbol{n}$ in the governing equation (45) ${ }_{2}$. The constraints (47) can also be used with the expanded form of $\boldsymbol{F}, \boldsymbol{n}$ and $\boldsymbol{G}$. Finally, substituting the expanded series of $\boldsymbol{S}$ in the bottom boundary condition $(46)_{1}$ yields

$$
\boldsymbol{S}^{(0)} \boldsymbol{k}=-\boldsymbol{q}^{-}(\boldsymbol{r}) \quad(\mathrm{Z}=0 \text { at the base })
$$

Similarly using series for $\boldsymbol{n}$ and $\tilde{\boldsymbol{T}}$, in the bottom boundary condition (46) $)_{3}$ yields

$$
\left(I-n^{(0)} \otimes n^{(0)}\right) \tilde{T}^{(0)} k=0
$$

Following the same procedure for the top boundary conditions results in an infinite series in $Z$. Therefore, we truncate the series s.t. our formulation is accurate upto $\mathcal{O}\left(h^{2}\right)$. [Liu et al., 2020] shows that the resulting formulation results in a closed system of 34 linear algebraic equations for 34 variables $\left(\boldsymbol{x}^{(0)} \ldots \boldsymbol{x}^{(4)}, \lambda_{1}^{(0)} \ldots \lambda_{1}^{(3)}, \boldsymbol{n}^{(0)} \ldots \boldsymbol{n}^{(4)}\right)$.

They further simplify the equations to get the final form

$$
\begin{array}{r}
\bar{\nabla} \cdot \overline{\boldsymbol{S}}=-\overline{\boldsymbol{q}} \\
\left(\boldsymbol{I}-\boldsymbol{n}^{(0)} \otimes \boldsymbol{n}^{(0)}\right) \overline{\boldsymbol{T}}+\boldsymbol{t}=0 \tag{53}
\end{array}
$$

with a constraint

$$
\begin{equation*}
n^{(0)} \cdot n^{(0)}=1 \tag{54}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{S}=\boldsymbol{S}^{(0)}+h \boldsymbol{S}^{(1)}+\frac{2}{3} h^{2} \boldsymbol{S}^{(2)}+\mathcal{O}\left(h^{3}\right), \quad \bar{T}=\tilde{\boldsymbol{T}}^{(0)}+h \tilde{\boldsymbol{T}}^{(1)}+\frac{2}{3} h^{2} \tilde{\boldsymbol{T}}^{(2)}+\mathcal{O}\left(h^{3}\right) \text { and } \boldsymbol{q}^{-}=\frac{\boldsymbol{q}^{+}+\boldsymbol{q}^{-}}{2 h} \tag{55}
\end{equation*}
$$

and term $\boldsymbol{t}$ is given in the Appendix B of [Liu et al., 2020]. The above set of equations can be transformed into a coupled fourth-order differential equations for $\boldsymbol{x}^{0}$ and $\boldsymbol{n}^{0}$ by eliminating all $\boldsymbol{S}^{(i)}$ and $\tilde{\boldsymbol{T}}^{(i)}$.

## Application to the pure finite-bending of an NLCE-substrate structure

The 3D system of governing equations (45)-(47) derived in the previous sections can be used to study the finite bending in a NLCE. In practice, the NLCE samples are very thin and hence it is more reasonable to study bending of a NLCE film bonded to a hyper-elastic material. The NLCE-substrate structure in both initial and current configuration is shown in the Fig. 3. The initial thickness of NLCE and substrate is equal to $2 h$ and $2 H$, respectively. The initial length of the sample is $2 L_{1}$ and and the initial width is $2 L_{2}$. The initial configuration is described using the Cartesian coordinate system ( $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ ) and current configuration using the cylindrical coordinate system $(r, y, \theta)$. During the deformation the interface remains intact and the plate deforms in a sector of cylindrical tube. From the Fig.3, the relation between the current and the reference configuration is given by

$$
\begin{gather*}
r=r(Z), \\
\theta=\frac{\alpha}{L_{1}} X,  \tag{56}\\
y=Y \leq 2 h \\
y=Y \leq L_{1} \leq X \leq 2 L_{2}
\end{gather*}
$$

where $\alpha$ is the bending angle of the plate. The deformation gradient tensor, $\boldsymbol{F}$, becomes

$$
\begin{align*}
\boldsymbol{F}=\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} & =\frac{\partial r \boldsymbol{e}_{r}}{\partial \boldsymbol{X}}+\frac{\partial y}{\partial \boldsymbol{X}} \boldsymbol{e}_{y} \\
& =\frac{\partial r}{\partial \boldsymbol{X}} \boldsymbol{e}_{r}+r(Z) \frac{\partial \boldsymbol{e}_{r}}{\partial \boldsymbol{X}}+\frac{\partial y}{\partial \boldsymbol{X}} \boldsymbol{e}_{y} \\
& =r^{\prime}(Z) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}+r(Z) \frac{\alpha}{L_{1}} \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}+\boldsymbol{e}_{y} \otimes \boldsymbol{E}_{2} \tag{57}
\end{align*}
$$

where we have used the identity

$$
\begin{equation*}
\frac{\partial \boldsymbol{a}(\theta)}{\partial \boldsymbol{X}}=\frac{d \boldsymbol{a}(\theta)}{d \theta} \otimes \frac{\partial \theta}{\partial \boldsymbol{X}} \tag{58}
\end{equation*}
$$

The substrate and the film both are in-compressible ans hence the incompressibility condition $\operatorname{det}(\boldsymbol{F})=1$ gives,

$$
\begin{align*}
\operatorname{det}(\boldsymbol{F}) & =\boldsymbol{F} \boldsymbol{E}_{1} \cdot\left(\boldsymbol{F} \boldsymbol{E}_{2} \times \boldsymbol{F} \boldsymbol{E}_{3}\right)  \tag{59}\\
& =\frac{\alpha}{L_{1}} r(Z) \boldsymbol{e}_{\theta} \cdot\left(\boldsymbol{e}_{y} \times r^{\prime}(Z) \boldsymbol{e}_{r}\right)  \tag{60}\\
& =\frac{\alpha}{L_{1}} r(Z) r^{\prime}(Z)=1 \tag{61}
\end{align*}
$$

The director $\boldsymbol{n}_{\mathbf{0}}$ in the initial configuration from Fig. 3 is along the Z-axis. We can also assume the director $\boldsymbol{n}$ to be in the plane of $\boldsymbol{e}_{r}, \boldsymbol{e}_{\theta}$, i.e.

$$
\begin{equation*}
\boldsymbol{n}=\sin \beta(Z) \boldsymbol{e}_{r}+\cos \beta(Z) \boldsymbol{e}_{\theta} \tag{62}
\end{equation*}
$$

We will use subscripts ' $f$ ' and ' $s$ ' to denote the properties of film and substrate, respectively. We assume the hyper-elastic substrate as in-compressible neo-Hookean material. The strain energy function of the substrate is then given by

$$
\begin{equation*}
\phi_{s}=\frac{\mu_{s}}{2}\left(\operatorname{tr}\left(\boldsymbol{F} \boldsymbol{F}^{T}\right)-3\right), \tag{63}
\end{equation*}
$$

and hence the nominal stress tensor $\boldsymbol{S}_{s}$ for the substrate, on using $(41)_{1}$, becomes

$$
\begin{equation*}
\boldsymbol{S}_{s}(\boldsymbol{F})=\mu_{s} \boldsymbol{F}-\lambda_{1 s} \boldsymbol{F}^{-T} \tag{64}
\end{equation*}
$$

The nominal stress expression in [Liu et al., 2021] is the transpose of the above equation.
The orientational gradient, $\boldsymbol{G}$, can be calculated using (58) and (62) as,

$$
\begin{align*}
\boldsymbol{G}=\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{X}} & =\frac{\partial}{\partial \boldsymbol{X}}\left(\sin \beta(Z) \boldsymbol{e}_{r}(\theta)\right)+\frac{\partial}{\partial \boldsymbol{X}}\left(\cos \beta(Z) \boldsymbol{e}_{\theta}(\theta)\right) \\
& =\cos \beta(Z) \beta^{\prime}(Z) \boldsymbol{e}_{r}(\theta) \otimes \boldsymbol{E}_{3}-\sin \beta(Z) \beta^{\prime}(Z) \boldsymbol{e}_{\theta}(\theta) \otimes \boldsymbol{E}_{3}+\sin \beta(Z) \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{e}_{r}(\theta)\right)+\cos \beta(Z) \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{e}_{\theta}(\theta)\right) \\
& =\cos \beta(Z) \beta^{\prime}(Z) \boldsymbol{e}_{r}(\theta) \otimes \boldsymbol{E}_{3}-\sin \beta(Z) \beta^{\prime}(Z) \boldsymbol{e}_{\theta}(\theta) \otimes \boldsymbol{E}_{3}+\sin \beta(Z) \frac{d \boldsymbol{e}_{r}(\theta)}{d \theta} \otimes \frac{\partial \theta}{\partial \boldsymbol{X}}+\cos \beta(Z) \frac{d \boldsymbol{e}_{\theta}(\theta)}{d \theta} \otimes \frac{\partial \theta}{\partial \boldsymbol{X}} \\
& =\cos \beta(Z) \beta^{\prime}(Z) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}-\sin \beta(Z) \beta^{\prime}(Z) \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{3}+\frac{\alpha}{L_{1}} \sin \beta(Z) \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}-\frac{\alpha}{L_{1}} \cos \beta(Z) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{1}, \tag{65}
\end{align*}
$$

We now have all quantities $(\boldsymbol{n}, \boldsymbol{F}, \boldsymbol{G})$ for the finite bending of NCLE-substrate in terms of the unknowns $r(Z), \beta(Z), \lambda_{1 f}(\boldsymbol{X})$ and $\lambda_{1 s}(\boldsymbol{X})$. We will assume that $\lambda_{1 f}$ and $\lambda_{1 s}$ are functions of $\mathbf{Z}$ only. The exact form of $\boldsymbol{S}_{f}, \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{\pi}}, \boldsymbol{S}_{s}$ can be calculated by using (57),(62) and (65) in (42)-(44) and(64). These forms are given in Appendix A of [Liu et al., 2021].

The incompressibility condition (58) is a simple ode for the variable $r(Z)$, which when solved gives

$$
\begin{array}{r}
\int \frac{\alpha}{L_{1}} r(Z) r^{\prime}(Z) d z=\int d Z \\
\int \frac{\alpha}{2 L_{1}} d\left(r^{2}\right)=\int d Z \\
r(Z)^{2}=\frac{2 L_{1} Z}{\alpha}+C_{1} \\
r(Z)=\sqrt{\frac{2 L_{1} Z}{\alpha}+C_{1}} \tag{66}
\end{array}
$$

[Liu et al., 2021] shows that the $\beta$ is independent of $Z$ and it can only be equal to $\pi / 2$ or 0 . The nominal stress tensor $\boldsymbol{S}_{s}$ in substrate can be found using

$$
\begin{aligned}
\boldsymbol{F}^{-T} & =\left(r^{\prime}(Z) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}+r(Z) \frac{\alpha}{L_{1}} \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}+\boldsymbol{e}_{y} \otimes \boldsymbol{E}_{2}\right)^{-T} \\
& =\lambda_{1 s}\left(r^{\prime}(Z) \boldsymbol{E}_{3} \otimes \boldsymbol{e}_{r}+r(Z) \frac{\alpha}{L_{1}} \boldsymbol{E}_{1} \otimes \boldsymbol{e}_{\theta}+\boldsymbol{E}_{2} \otimes \boldsymbol{e}_{y}\right)^{-1} \\
& =\left(r^{\prime}(Z) \cos \theta \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{3}+r^{\prime}(Z) \sin \theta \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{1}-r(Z) \frac{\alpha}{L_{1}} \sin \theta \boldsymbol{E}_{1} \otimes \boldsymbol{E}_{3}+r(Z) \frac{\alpha}{L_{1}} \cos \theta \boldsymbol{E}_{1} \otimes \boldsymbol{E}_{1}+\boldsymbol{E}_{2} \otimes \boldsymbol{E}_{2}\right)^{-1}, \\
& =\operatorname{det}(\boldsymbol{F})^{-1}\left(r^{\prime}(Z) \cos \theta \boldsymbol{E}_{1} \otimes \boldsymbol{E}_{1}-r^{\prime}(Z) \sin \theta \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{1}+r(Z) \frac{\alpha}{L_{1}} \sin \theta \boldsymbol{E}_{1} \otimes \boldsymbol{E}_{3}+r(Z) \frac{\alpha}{L_{1}} \cos \theta \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{3}+\boldsymbol{E}_{2} \otimes \boldsymbol{E}_{2}\right) \\
& =\left(r^{\prime}(Z) \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}+r(Z) \frac{\alpha}{L_{1}} \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}+\boldsymbol{e}_{y} \otimes \boldsymbol{E}_{2}\right)
\end{aligned}
$$

which gives

$$
\begin{align*}
\boldsymbol{S}_{s} & =\mu_{s} \boldsymbol{F}-\lambda_{1 s} \boldsymbol{F}^{-T} \\
& =\mu_{s}\left(r^{\prime}(Z) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}+r(Z) \frac{\alpha}{L_{1}} \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}+\boldsymbol{e}_{y} \otimes \boldsymbol{E}_{2}\right)-\lambda_{1 s}\left(r^{\prime}(Z) \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}+r(Z) \frac{\alpha}{L_{1}} \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}+\boldsymbol{e}_{y} \otimes \boldsymbol{E}_{2}\right) \\
& =\left(\mu_{s} r^{\prime}(Z)-\lambda_{1 s} r(Z) \frac{\alpha}{L_{1}}\right) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}+\left(\mu_{s} r(Z) \frac{\alpha}{L_{1}}-\lambda_{1 s} r^{\prime}(Z)\right) \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}+\left(\mu_{s}-\lambda_{1 s}\right) \boldsymbol{e}_{y} \otimes \boldsymbol{E}_{2} \tag{67}
\end{align*}
$$

The expression differs from that given in Appendix A of [Liu et al., 2021]. Substituting the above equation in $(45)_{1}$, we get the equilibrium condition for the substrate, which can be solved to get $\lambda_{1 s}$,

$$
\begin{array}{r}
\frac{d}{d Z}\left(\mu_{s} r^{\prime}(Z)-\lambda_{1 s} r(Z) \frac{\alpha}{L_{1}}\right)=0, \\
\frac{d}{d Z}\left(\mu_{s} \frac{L_{1}}{\alpha r(Z)}-\lambda_{1 s} r(Z) \frac{\alpha}{L_{1}}\right)=0, \\
\mu_{s} \frac{L_{1}}{\alpha r(Z)}-\lambda_{1 s} r(Z) \frac{\alpha}{L_{1}}=C_{2}, \\
\lambda_{1 s}=\mu_{s} \frac{L_{1}^{2}}{\alpha^{2} r(Z)^{2}}-\frac{C_{2} L_{1}}{\alpha r(Z)} \\
\lambda_{1 s}=\mu_{s} \frac{L_{1}^{2}}{\alpha^{2} C_{1}+2 L_{1} \alpha Z}-\frac{C_{2} L_{1}}{\left(2 L_{1} \alpha Z+\alpha^{2} C_{1}\right)^{1 / 2}} . \tag{68}
\end{array}
$$

For $\beta=\pi / 2$, the director in the current configuration $\boldsymbol{n}$ and the orientation gradient $\boldsymbol{G}$ simplifies to

$$
\boldsymbol{n}=\boldsymbol{E}_{3}, \quad \boldsymbol{G}=\frac{\alpha}{L_{1}} \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}
$$

Also following expressions simplifies as

$$
\begin{aligned}
\boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n} & =\boldsymbol{E}_{3} \otimes \boldsymbol{F}^{T} \boldsymbol{E}_{3}=r^{\prime}(Z)\left(\boldsymbol{e}_{r} \cdot \boldsymbol{E}_{3}\right) \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{3}+\frac{\alpha}{L_{1}} r(Z)\left(\boldsymbol{e}_{r} \cdot \boldsymbol{E}_{3}\right) \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{1}, \\
\boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0} & =\boldsymbol{F} \boldsymbol{E}_{3} \otimes \boldsymbol{E}_{3}=r^{\prime}(Z) \boldsymbol{e}_{r} \otimes \boldsymbol{E}_{3}, \\
\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0} & =\boldsymbol{F}^{T} \boldsymbol{E}_{3} \cdot \boldsymbol{E}_{3}=r^{\prime}(Z)\left(\boldsymbol{e}_{r} \cdot \boldsymbol{E}_{3}\right), \\
\boldsymbol{G} \boldsymbol{G}^{T} \boldsymbol{F} & =\left(\frac{\alpha}{L_{1}} \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1}\right)\left(\frac{\alpha}{L_{1}} \boldsymbol{E}_{1} \otimes \boldsymbol{e}_{\theta}\right) \boldsymbol{F}=\frac{\alpha^{2}}{L_{1}^{2}}\left(\boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{\theta}\right) \boldsymbol{F}=\frac{\alpha^{3}}{L_{1}^{3}} r(Z)^{2} \boldsymbol{e}_{\theta} \otimes \boldsymbol{E}_{1} .
\end{aligned}
$$

Using the above results, (57) and (42) in equilibrium equation $(45)_{1}$, we obtain

$$
\begin{aligned}
\operatorname{Div}\left(\boldsymbol{S}_{f}(\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{G})\right) & =\mu_{f}\left(\operatorname{Div}(\boldsymbol{F})-\operatorname{Div}\left(\frac{s-1}{s} \boldsymbol{n} \otimes \boldsymbol{F}^{T} \boldsymbol{n}+(s-1) \boldsymbol{F} \boldsymbol{n}_{0} \otimes \boldsymbol{n}_{0}-\frac{(s-1)^{2}}{s}\left(\boldsymbol{F}^{T} \boldsymbol{n} \cdot \boldsymbol{n}_{0}\right) \boldsymbol{n} \otimes \boldsymbol{n}_{0}\right)\right) \\
& +\operatorname{Div}\left(\frac{k(s-1)^{2}}{s} \boldsymbol{G}^{T} \boldsymbol{F}\right)-\operatorname{Div}\left(\lambda_{1 f} \boldsymbol{F}^{-T}\right) \\
& =\frac{d \mu_{f} r^{\prime}(Z)}{d Z} \boldsymbol{e}_{r}-\frac{d}{d Z}\left(\mu_{f} \frac{s-1}{s} r^{\prime}(Z) \cos \theta\right) \boldsymbol{E}_{3}+\frac{d}{d Z}\left(\mu_{f}(s-1) r^{\prime}(Z)\right) \boldsymbol{e}_{r} \\
& -\frac{d}{d Z}\left(\mu_{f} \frac{(s-1)^{2}}{s} r^{\prime}(Z) \cos \theta\right) \boldsymbol{E}_{3}-\frac{d}{d Z}\left(\frac{\alpha}{L_{1}} r(Z) \lambda_{1 f}\right) \boldsymbol{e}_{r},
\end{aligned}
$$

Taking dot product with $\boldsymbol{E}_{3}$ and equating to zero gives

$$
\begin{array}{r}
\frac{d}{d Z}\left(\mu_{f} r^{\prime}(Z)-\mu_{f} \frac{s-1}{s} r^{\prime}(Z)+\mu_{f}(s-1) r^{\prime}(Z)-\mu_{f} \frac{(s-1)^{2}}{s} r^{\prime}(Z)-\frac{\alpha}{L_{1}} r(Z) \lambda_{1 f}\right)=0 \\
\mu_{f} r^{\prime}(Z)\left(1-\frac{s-1}{s}+(s-1)-\frac{(s-1)^{2}}{s}\right)-\frac{\alpha}{L_{1}} r(Z) \lambda_{1 f}=C_{3} \\
\mu_{f} r^{\prime}(Z)-C_{3}=\frac{\alpha}{L_{1}} r(Z) \lambda_{1 f} \\
\lambda_{1 f}=\frac{L_{1}^{2}}{2 \alpha L_{1}+\alpha^{2} C_{1}}-\frac{L_{1} C_{3}}{\left(2 \alpha L_{1}+\alpha^{2} C_{1}\right)^{1 / 2}} \tag{69}
\end{array}
$$

Note that the form of $\lambda_{1 s}$ and $\lambda_{1 f}$ derived in (68) and(69), respectively, differs significantly from that given in [Liu et al., 2021](29,34). For $\beta=0$, similar calculation can be done to find $\lambda_{1 f}$. The constant $C_{1}, C_{2}$ and $C_{3}$ can be found using the boundary conditions.

## Conclusion

A variational formulation is used to derive the 3D equilibrium conditions for a NLCE plate. The energy density function proposed by [Anderson et al., 1999] is used for the strain energy. Since, NLCE samples have thickness much smaller than the other two dimensions, they can be treated as a 2D plate. The 2D equilibrium condition is derived using Taylor series expansion about the bottom of the plate. The 2D equation comes out to be a fourth order coupled differential equations. Finally, the 3D equilibrium equations are solved for the case of finite-bending of the NLCE-substrate plate. The final form of Lagrange multipliers $\lambda_{1 s}$ and $\lambda_{1 f}$ varies significantly from that given in [Liu et al., 2021].

## References

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## Figures



Figure 1: Polymers are on average spherical in the isotropic(I) state and elongate when they are cooled to the nematic (N) state. Source: [Warner and Terentjev, 2007]


Figure 2: The geometry of an NLCE plate in the reference configuration. The thickness of the plate is $2 h . \Omega$ represents the cross section of the plate. $\partial \Omega_{q}$ is the traction boundary condition where the traction is given by $\boldsymbol{q}$ and $\partial \Omega_{0}$ represents the displacement boundary condition. Source: [Liu et al., 2020]

(a) Reference configuration

(b) Current configuration

Figure 3: The figure shows the NLCE-substrate plate in reference and current configuration. The thickness of the substrate is $2 H$ and the thickness of the NLCE is $2 h$. The length of the plate is $2 L_{1}$ and the width of the plate is $2 L_{2}$. The Source: [Liu et al., 2021]

Tables

